V

THE STRESS TENSOR

1. The notion of stress

A BODY which is acted on by external forces, or, more generally, a body in which one part exerts a force on neighbouring parts, is said to be in a state of stress. If we consider a volume element situated within a stressed body, we may recognize two kinds of forces acting upon it.

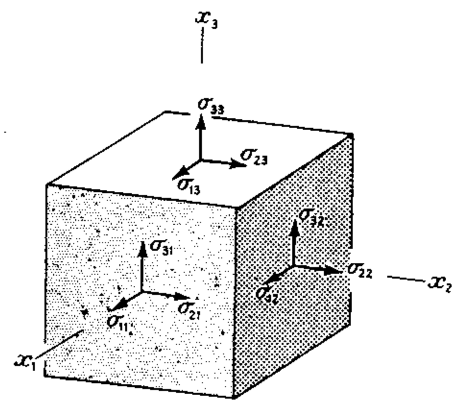


FIG. 5.1. The forces on the faces of a unit cube in a homogeneously stressed body.

First of all, there are body-forces, such as gravity, which act throughout the body on all its elements and whose magnitudes are proportional to the *volume* of the element. Secondly, there are forces exerted on the surface of the element by the material surrounding it. These forces are proportional to the *area* of the surface of the element, and the force per unit area is called the 'stress'. In the present chapter we discuss how this stress may be exactly specified. A stress is said to be *homogeneous* if the forces acting on the surface of an element of fixed shape and orientation are independent of the position of the element in the body.

1.1. Homogeneous stress. At first we confine the discussion to states in which (1) the stress is homogeneous throughout the body, (2) all parts of the body are in statical equilibrium, and (3) there are no body-forces or body-torques.

Consider a unit cube within the body (Fig. 5.1) with edges parallel

to the axes Ox_1, Ox_2, Ox_3. A force will be transmitted across each face of the cube, exerted by the material outside the cube upon the material inside the cube. The force transmitted across each face may be resolved into three components. Consider first the three faces which are towards the three positive ends of the axes (those shown in Fig. 5.1). We denote by σ_{ij} the component of force in the $+Ox_i$ direction transmitted across that face of the cube which is perpendicular to Ox_j.† Note the sign convention: σ_{12}, for example, is the force exerted in the $+Ox_1$ direction on the face normal to Ox_2, *by* the material outside the cube *upon* the material inside. Since the stress is homogeneous, the forces exerted on the cube across the three opposite faces must be equal and opposite to those shown in Fig. 5.1. σ_{11}, σ_{22}, σ_{33} are the *normal components* of stress and σ_{12}, σ_{21}, σ_{23} etc. are the *shear components*. In § 2 it is proved that the σ_{ij} thus defined form a second-rank tensor. With the sign convention we have adopted it will be seen that a positive value of σ_{11}, σ_{22} or σ_{33} implies a corresponding tensile stress; a negative value implies a compressive stress. This is the definition normally found in modern textbooks on elasticity. However, the opposite sign convention (compressive stresses positive) is sometimes used, particularly in applications to piezoelectricity and photoelasticity, and so care is needed when consulting numerical data given by different authors.

Our assumption (2), that the unit cube should be in statical equilibrium, imposes conditions on the σ_{ij}. Let us take moments about an axis parallel to Ox_1 passing through the centre of the cube (Fig. 5.2). Since the stress is homogeneous the three components of force on any face all pass through the mid-point of the face. The normal components and the shear components on the Ox_1 faces therefore give no moment, and we find as the condition for equilibrium

$$\sigma_{23} = \sigma_{32}.$$

In a similar way $\sigma_{31} = \sigma_{13}$ and $\sigma_{12} = \sigma_{21}$, and so we may write

$$\sigma_{ij} = \sigma_{ji}. \tag{1}$$

1.2. Inhomogeneous stress. The relation (1) continues to hold even when the stress is inhomogeneous, when the body is not in statical equilibrium, and when body-forces (but not body-torques) are present. This may be proved in the following way.

We define components of stress in essentially the same way as before,

† There is of course no connexion with electrical conductivity which we have also denoted by σ_{ij}.

but we now have to cover situations where the stress varies from point to point. The notion of the stress at a point is arrived at by a limiting process. We divide the force transmitted across a surface passing through the point by the area of the surface, and then let the area of the surface tend to zero at the point. Specifically, the component of force in the Ox_i direction transmitted across a surface element of area dS perpendicular to Ox_j is defined to be $\sigma_{ij}dS$ as dS tends to zero. The sign convention is the same as for homogeneous stress: namely,

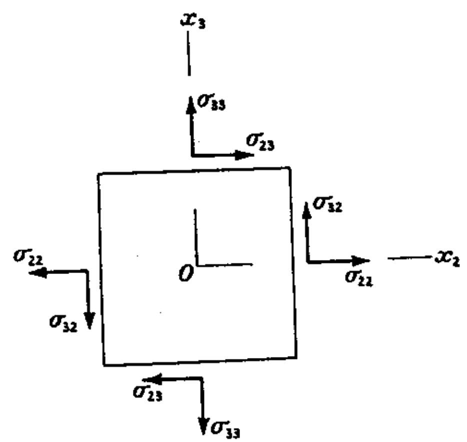


FIG. 5.2. The forces on the faces perpendicular to Ox_2 and Ox_3 of a unit cube in a homogeneously stressed body. The axis Ox_1 is normal to the plane of the figure.

$\sigma_{ij}dS$ denotes the force in the $+Ox_i$ direction exerted *by* the material on the $+Ox_j$ side of the element *upon* the material on the $-Ox_j$ side.

Consider now a small rectangular parallelepiped situated within the stressed body, centred on the origin, and with edges parallel to the axes and of lengths δx_1, δx_2, δx_3. Let σ_{ij} stand for the stresses at the origin. We wi h to find the equation of motion of the element in the Ox_1 direction. The average values of the component σ_{11} over the two faces perpendicular to Ox_1 are shown in Fig. 5.3. The forces in the Ox_1 direction on these two faces are

$$-\left(\sigma_{11}-\frac{\partial\sigma_{11}}{\partial x_1}\cdot\tfrac{1}{2}\delta x_1\right)\delta x_2\,\delta x_3 \quad\text{and}\quad \left(\sigma_{11}+\frac{\partial\sigma_{11}}{\partial x_1}\cdot\tfrac{1}{2}\delta x_1\right)\delta x_2\,\delta x_3,$$

so that the resultant is $$\frac{\partial\sigma_{11}}{\partial x_1}\delta x_1\,\delta x_2\,\delta x_3.$$

The forces in the Ox_1 direction on the two faces perpendicular to Ox_2 are

$$-\left(\sigma_{12}-\frac{\partial\sigma_{12}}{\partial x_2}\cdot\tfrac{1}{2}\delta x_2\right)\delta x_3\,\delta x_1 \quad\text{and}\quad \left(\sigma_{12}+\frac{\partial\sigma_{12}}{\partial x_2}\cdot\tfrac{1}{2}\delta x_2\right)\delta x_3\,\delta x_1,$$

with resultant $$\frac{\partial\sigma_{12}}{\partial x_2}\delta x_1\,\delta x_2\,\delta x_3.$$

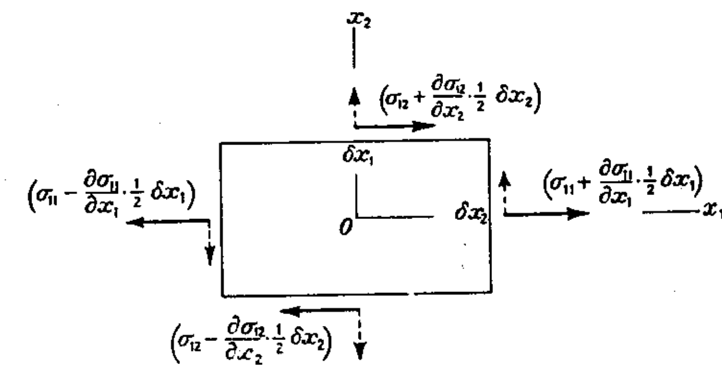


FIG. 5.3. Inhomogeneous stress: illustrating the forces on an element which have components parallel to Ox_1. The forces on the face perpendicular to Ox_3 are not shown.

Similarly, for the two faces perpendicular to Ox_3 we find

$$\frac{\partial\sigma_{13}}{\partial x_3}\delta x_1\,\delta x_2\,\delta x_3.$$

If there is a body-force with a component in the Ox_1 direction of g_1 per unit mass (gravity, for example), we have as the equation of motion

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}+\rho g_1=\rho\ddot{u}_1,$$

where ρ is the density and $\ddot{u}_1$ is the acceleration in the Ox_1 direction.

Resolution of forces in the Ox_2 and Ox_3 directions yields two similar equations. Thus, finally, we may write

$$\frac{\partial\sigma_{ij}}{\partial x_j}+\rho g_i=\rho\ddot{u}_i, \tag{2}$$

a fundamental equation which connects the spatial variation of stress in a body with the accelerations of its elements; it forms the starting-point for the study of elastic waves in solid bodies. (Note that we have introduced a new idea here: the differentiation of a second-rank tensor.) If all parts of the body are in statical equilibrium, equations (2) take the simpler form

$$\frac{\partial\sigma_{ij}}{\partial x_j}+\rho g_i=0, \tag{2 a}$$

known as the *equations of equilibrium* and much used in the theory of elasticity.

Let us now write down the equation of motion for rotation of the element about the Ox_1 axis. The couple (anticlockwise in Fig. 5.4) due to the shear components of stress on the two faces perpendicular to Ox_2 is

$$\left(\sigma_{32}-\frac{\partial\sigma_{32}}{\partial x_2}\cdot\tfrac{1}{2}\delta x_2\right)\delta x_1\,\delta x_3\cdot\tfrac{1}{2}\delta x_2+\left(\sigma_{32}+\frac{\partial\sigma_{32}}{\partial x_2}\cdot\tfrac{1}{2}\delta x_2\right)\delta x_1\,\delta x_3\cdot\tfrac{1}{2}\delta x_2 = \sigma_{32}\,\delta x_1\,\delta x_2\,\delta x_3.$$

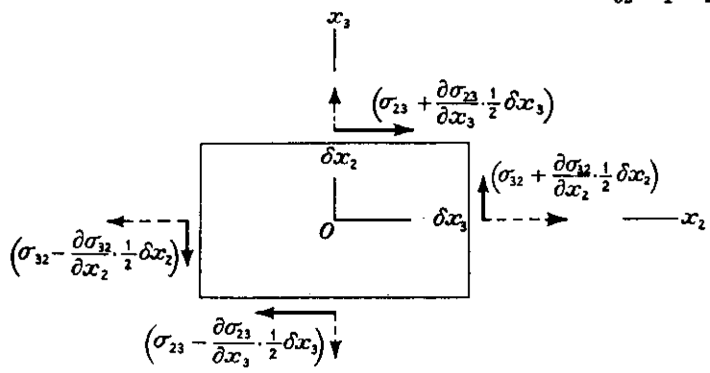


FIG. 5.4. Inhomogeneous stress: illustrating the forces on an element which have moments about the axis Ox_1.

The couple due to the shear components σ_{23} is similarly

$$-\sigma_{23}\,\delta x_1\,\delta x_2\,\delta x_3.\dagger$$

The equation of motion for rotation about Ox_1 is therefore

$$(\sigma_{32}-\sigma_{23})\delta x_1\,\delta x_2\,\delta x_3+G_1\,\delta x_1\,\delta x_2\,\delta x_3 = I_1\,\ddot{\theta}_1,$$

where I_1 is the moment of inertia, and $\ddot{\theta}_1$ the anticlockwise angular acceleration, about Ox_1. G_1 in this equation represents a body-torque. I_1 is of order of magnitude $\rho\,\delta x^5$. Hence, as the element becomes infinitesimally small, unless

$$\sigma_{32}-\sigma_{23}+G_1 = 0, \tag{3}$$

$\ddot{\theta}_1$ must increase without limit, as $1/\delta x^2$. In a continuous material such behaviour is impossible, and so we conclude that (3) holds.

† There will also be contributions from the normal components σ_{22} and σ_{33}, due to the fact that the resultant normal forces on the faces do not pass exactly through the mid-points of the faces. However, the lever arm of each of these forces will be an order of magnitude smaller than the lever arm for the shear forces and so the terms may be neglected. (These forces actually cancel in pairs to the first approximation and so their total contribution to the couple is an order of magnitude smaller still.) A similar consideration shows that we may also neglect the couple due to the slightly off-centre disposition of the shear components σ_{21} and σ_{31}.

The equations of motion for rotations about Ox_2 and Ox_3 respectively give in a similar way

$$\sigma_{13}-\sigma_{31}+G_2 = 0, \tag{4}$$

and

$$\sigma_{21}-\sigma_{12}+G_3 = 0. \tag{5}$$

A distributed body-torque, that is, a torque proportional to volume exerted by long-range forces, such as is represented by G_1, G_2, G_3, occurs when an anisotropic crystal becomes polarized or magnetized in a field, as we have seen in Chapter III, § 3 and Chapter IV, § 5†. Equations (3) to (5) show that the stress tensor σ_{ij} is then not symmetrical, but when body-torques are absent it is and we have simply

$$\sigma_{ij} = \sigma_{ji}. \tag{6}$$

Our treatment of elasticity (Ch. VIII) assumes (6), but in fact it is still perfectly valid even in the presence of body-torques (Tiffen and Stevenson 1956) provided that, in the statement of Hooke's law, σ_{ij} is interpreted as being not the stress tensor itself but its symmetrical part $\frac{1}{2}(\sigma_{ij}+\sigma_{ji})$. Similar considerations apply to piezoelectricity (Ch. VII) and photoelasticity (Ch. XIII). (See also pp. 315–317.)

2. Proof that the σ_{ij} form a tensor

We now prove that the components of stress σ_{ij} defined in §§ 1.1 and 1.2 form a second-rank tensor. We know (Ch. I, § 3) that if a set of quantities T_{ij} relate the components of two vectors p_i, q_i by an equation of the form

$$p_i = T_{ij}\,q_j,$$

the T_{ij} obey the tensor transformation law, and hence form a tensor. We accordingly prove that the σ_{ij} relate two vectors by an equation of this type.

Select any small surface element of area δS containing a point P within the stressed body. Draw a unit vector **l** perpendicular to it. Let the force transmitted across the area be denoted by $\mathbf{p}\,\delta S$ (Fig. 5.5). The force is taken to be that which is exerted by the material on the positive side of the area (defined by the direction of **l**) upon the material on the negative side. We have to ask: as **l** is altered in direction so that the surface element takes up different orientations, but always passing through P, how does $\mathbf{p}\,\delta S$ change? To answer this question we assume at first that the stress is homogeneous, that there are no body-forces, and that the body is in equilibrium. We consider the equilibrium of

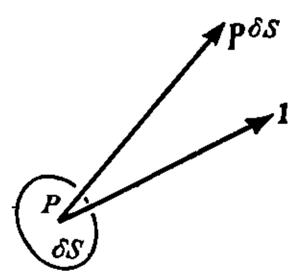


FIG. 5.5. The force transmitted across a small surface element δS in a stressed body.

† A small body-torque occurs during the phenomenon of optical activity (Ch. XIV).

the tetrahedron-shaped element of the body $OABC$ shown in Fig. 5.6. ABC represents our variable surface element perpendicular to **l** and the force transmitted across it is $\mathbf{p}\times$(area ABC). The forces on the three

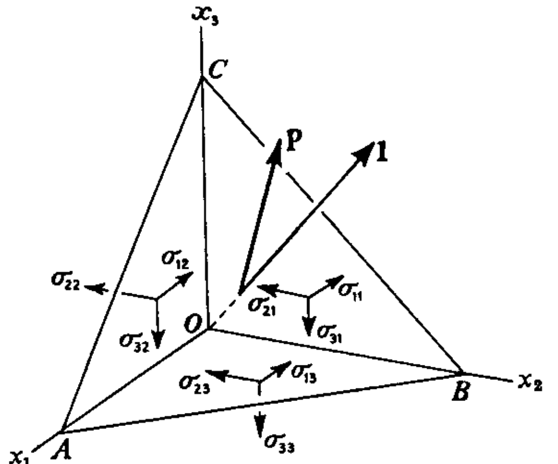


FIG. 5.6. The surface tractions on the faces of a tetrahedron bounded by the face ABC and the three coordinate planes.

faces at right angles may be specified by the stress components σ_{ij}, as shown. Resolving forces parallel to Ox_1 we have

$$p_1.ABC = \sigma_{11}.BOC+\sigma_{12}.AOC+\sigma_{13}.AOB,$$

or $\qquad p_1 = \sigma_{11}l_1+\sigma_{12}l_2+\sigma_{13}l_3.$

Similarly, $\qquad p_2 = \sigma_{21}l_1+\sigma_{22}l_2+\sigma_{23}l_3,$

and $\qquad p_3 = \sigma_{31}l_1+\sigma_{32}l_2+\sigma_{33}l_3.$

Hence, we may write $\qquad p_i = \sigma_{ij}l_j. \qquad (7)$

When the stress is not homogeneous, when body-forces are acting, and when the body is not in statical equilibrium, equations (7) still hold for any given point, for it is easy to see that the extra terms that enter become negligible as the tetrahedron is made vanishingly small.

Since σ_{ij} relates the two vectors p_i and l_j in a linear way it is a tensor. Equation (6) shows that it is a symmetrical tensor, and consequently it may be referred to its principal axes (Ch. I, § 4.1). Thus, on transformation to the principal axes, we have

$$\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{31}\\ \sigma_{12} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{23} & \sigma_{33}\end{bmatrix} \to \begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix};$$

σ_1, σ_2, σ_3 are the *principal stresses*.

When the principal stress directions are chosen as axes the shear stress components vanish, and the forces on the faces of a unit cube cut from the body with edges parallel to the axes are as shown in Fig. 5.7.

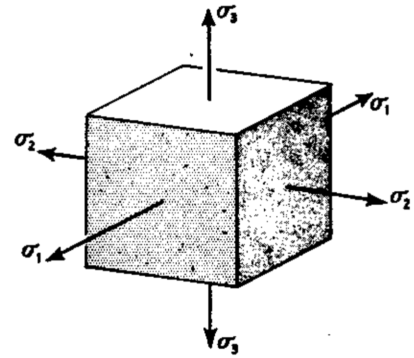


FIG. 5.7. The forces on the faces of a unit cube cut with its edges parallel to the three principal stress directions.

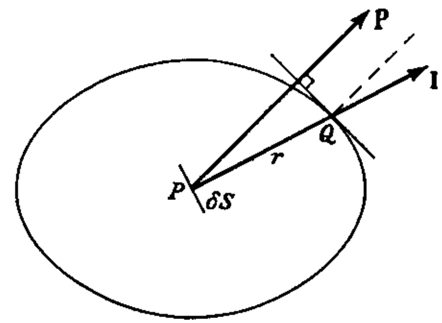


FIG. 5.8. Illustrating how the direction of the resultant force $\mathbf{p}\,\delta S$ transmitted across a small area δS may be found by using the radius-normal property of the stress quadric. **l** is the unit vector normal to δS. **p** is perpendicular to the tangent plane to the quadric at Q. The plane of the figure is that central cross-section of the quadric which contains both **p** and **l**. The element δS and the tangent plane are seen edge on.

3. The stress quadric

The representation quadric for σ_{ij} (Ch. I, § 4) is called simply the *stress quadric*. Its equation is $\qquad \sigma_{ij}x_ix_j = 1,$

or, referred to principal axes,

$$\sigma_1x_1^2+\sigma_2x_2^2+\sigma_3x_3^2 = 1.$$

The lengths of the semi-axes are therefore $1/\sqrt{\sigma_1}$, $1/\sqrt{\sigma_2}$, $1/\sqrt{\sigma_3}$. Since σ_1, σ_2, σ_3 may each be positive or negative, the quadric may be a real or imaginary ellipsoid or a hyperboloid.

The direction of the resultant force $\mathbf{p}\,\delta S$ transmitted across a small area δS may be found from the stress quadric by the radius-normal property (p. 28). Draw (Fig. 5.8) a radius vector of length r parallel to **l**, the unit vector normal to δS. Let it cut the surface of the quadric in Q. Then **p** is parallel to the normal to the quadric at Q. Whenever Q lies on one of the three principal axes, **p** is parallel to **l**; that is, there are no shear components.

By the property described on pp. 26, 27, the length of the radius vector r gives the normal stress σ transmitted across the element in Fig. 5.8:

thus $\sigma = 1/r^2$. Alternatively, σ is given analytically (Ch. I, § 6.2) by

$$\sigma = \sigma_{ij} l_i l_j;$$

or, if all components are referred to the principal stress axes,

$$\sigma = \sigma_1 l_1^2 + \sigma_2 l_2^2 + \sigma_3 l_3^2.$$

4. Special forms of the stress tensor

We give now some of the forms taken by the stress tensor, referred to its principal axes, in special cases.

(i) *Uniaxial stress*, σ.

$$\begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

An example is the stress in a long, vertical rod loaded by hanging a weight on the end. A non-uniform distribution of uniaxial stress occurs in the pure bending of a long bar.

(ii) *Biaxial stress*.

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

An example of a non-uniform distribution of biaxial stress is the stress

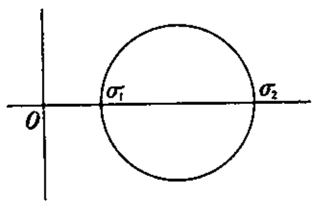


FIG. 5.9. The Mohr circle for a state of biaxial stress.

in a thin plate loaded by forces and couples applied to its edges. The Mohr circle for a state of biaxial stress is shown in Fig. 5.9.

(iii) *Triaxial stress*. This is an alternative name for the most general stress system with three non-zero principal stresses.

(iv) *Hydrostatic pressure*, p.

$$\begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \text{ or } -p\delta_{ij}.$$

(v) *Pure shear stress*.

$$\begin{bmatrix} -\sigma & 0 & 0 \\ 0 & \sigma & 0 \end{bmatrix}.$$

This is a special case of biaxial stress. A non-uniform distribution of pure shear stress occurs in a long rod subjected to pure torsion. The Mohr circle construction, illustrated for this case in Fig. 5.10*a*, shows at once that if the axes are turned through 45° about Ox_3 the normal stresses vanish (hence the name 'pure shear stress') and the tensor takes the form

$$\begin{bmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Ox_3 is the *axis of shear*. Figs. 5.10*b*, *c* show the forces acting on the faces of two elements in the two different orientations.

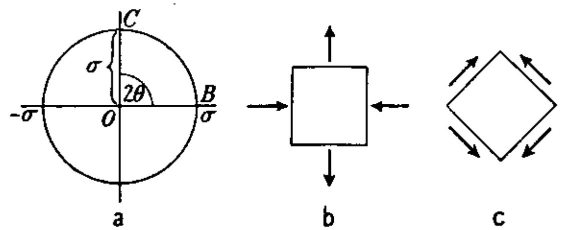


FIG. 5.10. Pure shear stress: (*a*) the Mohr circle, (*b*) and (*c*) the forces on an element in the orientations corresponding to points B and C respectively.

5. Difference between the stress tensor and tensors representing crystal properties

We conclude this chapter by pointing out an important distinction between the stress tensor and all the other second-rank tensors so far introduced. Tensors which measure crystal properties (such as the permittivity and the magnetic susceptibility, represented by quadrics) have definite orientations within a crystal, and, as we have seen, they must conform to the crystal symmetry. They are called *matter tensors*. The stress tensor, on the other hand, in common with the strain tensor of the next chapter, can have any orientation within a crystal, and it can exist just as well in isotropic bodies like glass as in anisotropic crystals. The stress tensor does not represent a crystal property but is akin to a 'force' impressed on the crystal; in this respect it is like an electric field, which can, of course, have an arbitrary direction in a crystal. Such tensors are called *field tensors*.

SUMMARY

THE stress at a point P in a material may be defined in the following way (§ 2). Let δS be the area of an element of surface passing through P. Draw a unit vector $\mathbf{l}$ perpendicular to it. Let the force transmitted across the area be $\mathbf{p}\,\delta S$ in the following sense: $\mathbf{p}\,\delta S$ is the force exerted by the material on the positive side of the area (defined by the direction of $\mathbf{l}$) upon the material on the negative side. Then statical considerations (or dynamical considerations if the body is not in statical equilibrium) show that, as $\delta S \to 0$, $\mathbf{p}$ is connected with $\mathbf{l}$ by the relation

$$p_i = \sigma_{ij} l_j,$$

where the σ_{ij} are coefficients. $[\sigma_{ij}]$ is a second-rank tensor and is called the *stress* at the point.

The meanings of the nine components σ_{ij} may be appreciated by considering the forces on the faces of a cube within the stressed body with edges parallel to the axes x_i (Fig. 5.1). The components with $i = j$ are the *normal components* of stress (positive for tensile stresses) and the components with $i \neq j$ are the *shear components* (§ 1.1).

The *equations of motion for translation* of a small element are (§ 1.2)

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = \rho \ddot{u}_i,$$

or if all parts of the body are in statical equilibrium,

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = 0,$$

where ρ is the density and g_i is the body-force per unit mass.

The *equations of motion for rotation* are (§ 1.2)

$$\left.\begin{aligned} \sigma_{32} - \sigma_{23} + G_1 &= 0 \\ \sigma_{13} - \sigma_{31} + G_2 &= 0 \\ \sigma_{21} - \sigma_{12} + G_3 &= 0 \end{aligned}\right\},$$

where G_i is the body-torque per unit volume. Hence, in the absence of body-torques,

$$\sigma_{ij} = \sigma_{ji}.$$

Since $[\sigma_{ij}]$ is a symmetrical tensor (in the absence of body-torques), it may be referred to its principal axes (§ 2) and may be represented by a quadric, $\sigma_{ij} x_i x_j = 1$. Stress is not a crystal property like the other second-rank tensors (magnetic susceptibility, permittivity, etc.) introduced so far (§ 5); it is akin to a 'force' impressed on the crystal. Accordingly, the stress quadric does not have to conform to the crystal symmetry.

EXERCISE 5.1. Show that a general stress may, by a suitable choice of axes, be expressed as the sum of (1) a hydrostatic stress (i.e. of the form $\sigma \delta_{ij}$) and (2) a shear stress (i.e. a stress whose normal components are all zero).

VI

THE STRAIN TENSOR AND THERMAL EXPANSION

THE problem of specifying the state of deformation of a solid body, which we take up in this chapter, may be approached by considering first the simpler one-dimensional and two-dimensional cases.

1. One-dimensional strain

Fig. 6.1 a shows an extendible string. We mark an origin O, fixed in space, and then stretch the string. After stretching (Fig. 6.1 b), an arbitrary point P on the string moves to P'. Let

$$OP = x \quad \text{and} \quad OP' = x+u.$$

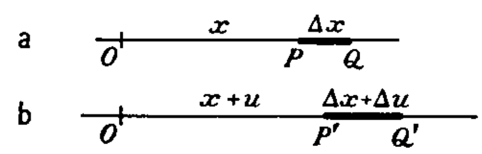


FIG. 6.1. The deformation of an extendible string: (a) unstretched, (b) stretched.

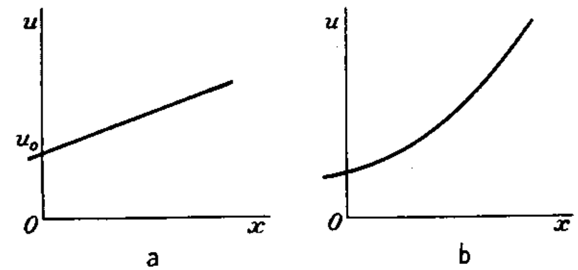


FIG. 6.2. The displacement u as a function of x in an extended string; (a) homogeneous stretching, (b) inhomogeneous stretching.

The variation of the displacement u with x is shown in Figs. 6.2 a and b. In Fig. 6.2 a, where u is a linear function of x, the string is stretched homogeneously; Fig. 6.2 b illustrates the more general case of inhomogeneous stretching. Let a point Q, close to P, move to Q' during the stretching and let $PQ = \Delta x$. Then $P'Q' = \Delta x + \Delta u$. In studying strain we are not concerned with the actual displacement of points but with

SUMMARY

Definition of a third-rank tensor. A set of 27 numbers T_{ijk} which represent a physical quantity are said to form a third-rank tensor if they transform according to the equation

$$T'_{ijk} = a_{il}\,a_{jm}\,a_{kn}\,T_{lmn}.$$

This is the same as the transformation equation for products of coordinates $x_i\,x_j\,x_k$.

The direct piezoelectric effect is given by

$$P_i = d_{ijk}\,\sigma_{jk}, \tag{3}$$

where the d_{ijk} are the piezoelectric moduli; they form a third-rank tensor.

Since σ_{jk} is the symmetric part of the stress tensor we may put $d_{ijk} = d_{ikj}$. This reduces the number of independent d_{ijk} to 18.

Matrix notation. The second and third suffixes in d_{ijk}, and both suffixes of σ_{jk}, are abbreviated into a single suffix running from 1 to 6, thus:

tensor notation	11	22	33	23, 32	31, 13	12, 21
matrix ,,	1	2	3	4	5	6

We also introduce factors of 2 as follows:

$$d_{ijk} = d_{in} \quad \text{when } n = 1, 2 \text{ or } 3,$$

and

$$2d_{ijk} = d_{in} \quad \text{when } n = 4, 5 \text{ or } 6.$$

We may then write (3) as

$$P_i = d_{ij}\,\sigma_j \quad (i = 1,2,3;\, j = 1,2,\ldots,6). \tag{17}$$

The coefficients d_{ij} form a 3-row, 6-column, matrix (d_{ij}).

The converse piezoelectric effect is given by

$$\epsilon_{jk} = d_{ijk}\,E_i, \tag{19}$$

with the same d_{ijk} as in the direct effect. The two suffixes in the ϵ_{jk} are abbreviated into a single one according to the above scheme by writing $\epsilon_{jk} = \epsilon_n$ when $n = 1$, 2 or 3 and $2\epsilon_{jk} = \epsilon_n$ when $n = 4$, 5 or 6. (19) is then written

$$\epsilon_j = d_{ij}\,E_i \quad (i = 1,2,3;\, j = 1,2,\ldots,6). \tag{21}$$

Symmetry requirements. A crystal with a centre of symmetry cannot be piezoelectric. The restrictions imposed by crystal symmetry on the moduli d_{ij} in the non-centrosymmetrical crystal classes may often be obtained by pure symmetry arguments. The *direct inspection method* provides a systematic approach, except for the trigonal and hexagonal systems, where it is necessary to use a more comprehensive analytical method. Group theory may also be used for this purpose.

A third-rank tensor such as d_{ijk} cannot be completely represented by a single surface. The *longitudinal piezoelectric surface* represents the component of polarization developed parallel to an applied tensile stress. The radius vector in any direction is directly proportional to the longitudinal effect in that direction. A longitudinal piezoelectric effect can only occur along a polar direction.

GENERAL REFERENCES

The two most comprehensive books on piezoelectricity are (see also pp. 313–19):

CADY, W. G. (1946) *Piezoelectricity*, New York: McGraw-Hill.
MASON, W. P. (1950) ***Piezoelectric crystals and their applications to ultrasonics***, **New York: van Nostrand.**

VIII

ELASTICITY. FOURTH-RANK TENSORS

1. Hooke's Law

A SOLID body changes its shape when subjected to a stress. Provided the stress is below a certain limiting value, the *elastic limit*, the strain is recoverable, that is to say, the body returns to its original shape when the stress is removed. It is further observed (Hooke's Law) that for sufficiently small stresses the amount of strain is proportional to the magnitude of the applied stress. For example, suppose a bar of an isotropic solid is loaded in pure tension so that the tensile stress is σ. The longitudinal strain ϵ equals $\Delta l/l$, where Δl is the increase in length and l is the original length. Hooke's Law states that

$$\epsilon = s\sigma,$$

where s is a constant. s is called the *elastic compliance constant* or, shortly, the *compliance*, for this particular arrangement of stress and strain directions. As an alternative we could write

$$\sigma = c\epsilon, \qquad c = 1/s,$$

where c is the *elastic stiffness constant*, or the *stiffness*. c is also Young's Modulus.†

These statements and definitions must now be generalized. We have seen (Chs. V and VI) that a homogeneous stress and a homogeneous strain are each specified, in general, by second-rank tensors. It is found that, if a general homogeneous stress σ_{ij} is applied to a crystal, the resulting homogeneous strain ϵ_{ij} is such that each component is linearly

† The use of the terms *compliance* and *stiffness* accords with current American usage. The corresponding terms used by many English authors are, respectively, *elastic modulus* and *elastic constant*. If this latter system is adopted we have the confusing fact that Young's Modulus is not an elastic modulus but an elastic constant. Another reason for preferring the words 'compliance' and 'stiffness' is that they are descriptive: a stiff crystal has a high value of its *stiffness*, a compliant crystal has a high value of its *compliance*. It is unfortunate, but a fact easily memorized, that the initial letters of 'stiffness' and 'compliance' are the reverse of the corresponding symbols.

Symbol	*American authors and this book*	*English authors*	*Dimensions*
s	compliance	modulus	stress^{-1}
c	stiffness	constant	stress

related to all the components of the stress. Thus, for example,

$$\begin{aligned}\epsilon_{11} = \quad & s_{1111}\sigma_{11}+s_{1112}\sigma_{12}+s_{1113}\sigma_{13}+ \\ & +s_{1121}\sigma_{21}+s_{1122}\sigma_{22}+s_{1123}\sigma_{23}+ \\ & +s_{1131}\sigma_{31}+s_{1132}\sigma_{32}+s_{1133}\sigma_{33},\end{aligned}$$

and eight similar equations for the other eight components of ϵ_{ij}, where the s's are constants. The generalized form of Hooke's Law may therefore be written E^{-1}

$$\epsilon_{ij} = s_{ijkl}\sigma_{kl}; \tag{1}$$

the s_{ijkl} are the *compliances* of the crystal. Equation (1) stands for nine equations, each with nine terms on the right-hand side. There are 81 s_{ijkl} coefficients.

If we apply only one component of stress, say σ_{11}, equations (1) imply that all the strain components, not just ϵ_{11}, may be different from zero. It follows that, if a rectangular block of crystal is loaded by a uniaxial tension applied parallel to one set of edges, it will not only stretch in the direction of the tension but it may also shear so that all the angles between the edges become different from right angles. As a corollary, if we try to bend a bar of crystal by applying pure bending couples to its ends, it will, in general, twist as well as bend. Correspondingly, if we try to twist a rod of crystal by applying pure twisting couples to its ends it will, in general, bend as well as twist.

As an alternative to equations (1) the stresses may be expressed in terms of the strains by the equations,

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}, \tag{2}$$

where the c_{ijkl} are the 81 *stiffness constants* of the crystal. If the relations (1) were solved as a set of simultaneous equations for the σ_{ij}, a set of solutions of the form (2) would be obtained, the coefficients c_{ijkl} being functions of the s_{ijkl}. The form of the relation between the c_{ijkl} and the s_{ijkl} is treated in § 7 and in Chapter IX, § 4.2.

The physical meaning of the s_{ijkl} may be appreciated by imagining the crystal to be subjected to various simple stress conditions. We recall from p. 87 that σ_{ij} may always be taken as symmetrical, even when body-torques are present. Hence, if a shear stress about Ox_3 were applied, both σ_{12} and σ_{21} would be present and we should have

$$\epsilon_{11} = s_{1112}\sigma_{12}+s_{1121}\sigma_{21} = (s_{1112}+s_{1121})\sigma_{12}.$$

s_{1112} and s_{1121} always occur together; it follows that it is in principle impossible to devise an experiment by which s_{1112} can be separated from s_{1121} and, in general, by which s_{ijkl} can be separated from s_{ijlk}. A similar situation arose with the piezoelectric moduli in

Chapter VII, § 1. Therefore, to avoid an arbitrary constant we set the two components equal:

$$s_{ijkl} = s_{ijlk}. \tag{3}$$

If, on the other hand, a uniaxial tension were applied parallel to Ox_3 the components of strain would be given by

$$\epsilon_{11} = s_{1133}\sigma_{33}, \quad \epsilon_{22} = s_{2233}\sigma_{33}, \quad \text{etc.}$$

In particular,

$$\epsilon_{12} = s_{1233}\sigma_{33} \quad \text{and} \quad \epsilon_{21} = s_{2133}\sigma_{33}.$$

But, from the definition of the components of the strain tensor (Ch. VI), $\epsilon_{12} = \epsilon_{21}$. Hence, $s_{1233} = s_{2133}$, and, in general, by considering other special cases, we see that

$$s_{ijkl} = s_{jikl}. \tag{4}$$

On account of the relations (3) and (4), only 36 of the 81 components s_{ijkl} are independent.

To attach physical meanings to the c_{ijkl} in equation (2) we have to imagine a set of stress components applied to the crystal and chosen in such a way that all the components of strain, except for one normal component or a pair of shear components, vanish. Thus the requisite stresses to produce tensor shear strain components ϵ_{12}, ϵ_{21} are

$$\sigma_{ij} = c_{ij12}\epsilon_{12}+c_{ij21}\epsilon_{21} = (c_{ij12}+c_{ij21})\epsilon_{12}.$$

Again we put the pairs of coefficients that always occur together equal to one another. Then, in general,

$$c_{ijkl} = c_{ijlk}. \tag{5}$$

By considering special cases, as with the s_{ijkl}, we also find that

$$c_{ijkl} = c_{jikl}. \tag{6}$$

Again, the equations (5) and (6) reduce the number of independent c_{ijkl} from 81 to 36.

We shall now show that the 81 compliances s_{ijkl} form a *fourth-rank tensor*. A fourth-rank tensor is defined, like tensors of lower rank, by its transformation law. The 81 numbers T_{ijkl} representing a physical quantity are said to form a fourth-rank tensor if they transform on change of axes to T'_{ijkl}, where

$$T'_{ijkl} = a_{im}a_{jn}a_{ko}a_{lp}T_{mnop}. \tag{7}$$

To prove that the s_{ijkl} form such a tensor we proceed as follows. We have

$$\epsilon'_{ij} = a_{ik}a_{jl}\epsilon_{kl}, \tag{8}$$

$$\epsilon_{kl} = s_{klmn}\sigma_{mn}, \tag{9}$$

$$\sigma_{mn} = a_{om}a_{pn}\sigma'_{op}. \tag{10}$$

Hence, combining these three equations, which form the scheme

$$\epsilon' \xrightarrow{(8)} \epsilon \xrightarrow{(9)} \sigma \xrightarrow{(10)} \sigma',$$

where $\rightarrow$ means 'in terms of', we obtain

$$\epsilon'_{ij} = a_{ik}\,a_{jl}\,s_{klmn}\,a_{om}\,a_{pn}\,\sigma'_{op}.$$

But we have

$$\epsilon'_{ij} = s'_{ijop}\,\sigma'_{op},$$

and so, by comparing coefficients,

$$s'_{ijop} = a_{ik}\,a_{jl}\,a_{om}\,a_{pn}\,s_{klmn}.$$

On interchanging the dummy suffixes this becomes

$$s'_{ijkl} = a_{im}\,a_{jn}\,a_{ko}\,a_{lp}\,s_{mnop}, \tag{11}$$

which is the necessary transformation law. It is worth noting, as a reminder of the economy of the dummy suffix notation, that equation (11) typifies 3^4 equations each with 3^4 terms on the right-hand side, making a total of $3^8 = 6{,}561$ terms in all.

The above proof is a general one. If two second-rank tensors A_{ij} and B_{kl} are related by the equation

$$A_{ij} = C_{ijkl}\,B_{kl},$$

the quantities C_{ijkl} form a fourth-rank tensor. It follows that the elastic stiffness constants c_{ijkl} also form a fourth-rank tensor.

2. The matrix notation

The symmetry of s_{ijkl} and c_{ijkl} in the first two and the last two suffixes makes it possible to use the matrix notation introduced in the preceding chapter. Both the stress components and the strain components are written, as before, with a single suffix running from 1 to 6:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix}, \qquad \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{31} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} \rightarrow \begin{bmatrix} \epsilon_1 & \frac{1}{2}\epsilon_6 & \frac{1}{2}\epsilon_5 \\ \frac{1}{2}\epsilon_6 & \epsilon_2 & \frac{1}{2}\epsilon_4 \\ \frac{1}{2}\epsilon_5 & \frac{1}{2}\epsilon_4 & \epsilon_3 \end{bmatrix}. \tag{12}$$

In the s_{ijkl} and the c_{ijkl} the first two suffixes are abbreviated into a single one running from 1 to 6, and the last two are abbreviated in the same way, according to the scheme,

tensor notation	11	22	33	23, 32	31, 13	12, 21
matrix notation	1	2	3	4	5	6

At the same time factors of 2 and 4 are introduced as follows:

$s_{ijkl} = s_{mn}$ when m and n are 1, 2 or 3,

$2s_{ijkl} = s_{mn}$ when either m or n are 4, 5 or 6,

$4s_{ijkl} = s_{mn}$ when both m and n are 4, 5 or 6.

Now consider equation (1) written out for ϵ_{11} and ϵ_{23}:

$$\begin{aligned} \epsilon_{11} &= s_{1111}\,\sigma_{11}+s_{1112}\,\sigma_{12}+s_{1113}\,\sigma_{13}+ \\ &\quad +s_{1121}\,\sigma_{21}+s_{1122}\,\sigma_{22}+s_{1123}\,\sigma_{23}+ \\ &\quad +s_{1131}\,\sigma_{31}+s_{1132}\,\sigma_{32}+s_{1133}\,\sigma_{33}; \end{aligned} \qquad \begin{aligned} \epsilon_{23} &= s_{2311}\,\sigma_{11}+s_{2312}\,\sigma_{12}+s_{2313}\,\sigma_{13}+ \\ &\quad +s_{2321}\,\sigma_{21}+s_{2322}\,\sigma_{22}+s_{2323}\,\sigma_{23}+ \\ &\quad +s_{2331}\,\sigma_{31}+s_{2332}\,\sigma_{32}+s_{2333}\,\sigma_{33}. \end{aligned}$$

In the matrix notation these two equations become

$$\begin{aligned} \epsilon_1 = {} & s_{11}\,\sigma_1+\tfrac{1}{2}s_{16}\,\sigma_6+\tfrac{1}{2}s_{15}\,\sigma_5+ \\ & +\tfrac{1}{2}s_{16}\,\sigma_6+ s_{12}\,\sigma_2+\tfrac{1}{2}s_{14}\,\sigma_4+ \\ & +\tfrac{1}{2}s_{15}\,\sigma_5+\tfrac{1}{2}s_{14}\,\sigma_4+ s_{13}\,\sigma_3; \end{aligned} \qquad \begin{aligned} \tfrac{1}{2}\epsilon_4 = {} & \tfrac{1}{2}s_{41}\,\sigma_1+\tfrac{1}{4}s_{46}\,\sigma_6+\tfrac{1}{4}s_{45}\,\sigma_5+ \\ & +\tfrac{1}{4}s_{46}\,\sigma_6+\tfrac{1}{2}s_{42}\,\sigma_2+\tfrac{1}{4}s_{44}\,\sigma_4+ \\ & +\tfrac{1}{4}s_{45}\,\sigma_5+\tfrac{1}{4}s_{44}\,\sigma_4+\tfrac{1}{2}s_{43}\,\sigma_3; \end{aligned}$$

or

$$\epsilon_1 = s_{1j}\,\sigma_j \quad \text{and} \quad \epsilon_4 = s_{4j}\,\sigma_j.$$

In general, therefore, equation (1) takes the shorter form

$$\epsilon_i = s_{ij}\,\sigma_j \quad (i,j = 1,2,\ldots,6). \tag{13}$$

The reason for introducing the 2's and 4's into the definitions of the s_{ij} is to avoid the appearance of 2's and 4's in equation (13) and to make it possible to write this equation in a compact form.†

For the c_{ijkl} no factors of 2 or 4 are necessary. For if we write simply

$$c_{ijkl} = c_{mn} \quad (i,j,k,l = 1,2,3;\ m,n = 1,\ldots,6),$$

it may be shown by writing out some typical members that equations (2) take the form

$$\sigma_i = c_{ij}\,\epsilon_j \quad (i,j = 1,2,\ldots,6). \tag{14}$$

The arrays of s_{ij} and c_{ij} written out in squares, thus:

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{pmatrix} \text{ and } \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{pmatrix} \tag{15}$$

are matrices, (s_{ij}) and (c_{ij}). As with the piezoelectric moduli we add the reminder that, in spite of their appearance with two suffixes, the s_{ij} and c_{ij} are not the components, and so do not transform like the components, of a second-rank tensor. To transform them to other axes it is necessary to go back to the tensor notation.

† The placing of the 2's and 4's in the definitions of the s_{ij} rather than in equation (13) conforms to established practice, to the recommendations in *Standards on piezoelectric crystals* (1949), and to the usage introduced by Voigt (1910). Wooster (1938) adopts the reverse procedure. Because of the two possible definitions care is needed when looking up numerical values.

3. The energy of a strained crystal

Consider a crystal which in the unstrained state has the form of a unit cube, and suppose it is subjected to a small homogeneous strain with components ϵ_i. Now let the strain components all be changed to $\epsilon_i+d\epsilon_i$. We prove that the work done by the stress components σ_i acting on the cube faces is

$$dW = \sigma_i\, d\epsilon_i \quad (i = 1, 2,..., 6). \tag{16}$$

First suppose that the strain component ϵ_1 is increased to $\epsilon_1+d\epsilon_1$, while the other strain components, and the position of the centre of the cube, remain unaltered. The two faces perpendicular to Ox_1 will move outwards by amounts $\frac{1}{2}d\epsilon_1$; the other four faces will simply increase in area, but the positions of their centres will be unchanged. The work done by the forces on these last four faces is therefore zero. The work done on the faces perpendicular to Ox_1 equals their displacement multiplied by the normal component of the force on them; it is therefore $2\sigma_1.\frac{1}{2}d\epsilon_1 = \sigma_1\, d\epsilon_1$. This is the term with $i = 1$ in (16); the terms with $i = 2$ and $i = 3$ are obtained in a similar way.

Now let the cube be sheared by making the two faces perpendicular to Ox_2 move in opposite directions parallel to Ox_3, so as to increase the strain component ϵ_4 to $\epsilon_4+d\epsilon_4$. In this deformation (simple shear) the mid-points of the faces perpendicular to Ox_2 each move a distance $\frac{1}{2}d\epsilon_4$. The component of force on the faces in this direction is σ_4. The work done by the forces is therefore $2\sigma_4.\frac{1}{2}d\epsilon_4 = \sigma_4\, d\epsilon_4$. The terms with $i = 5$ and $i = 6$ in (16) are obtained in a similar way.

It is readily shown that the corresponding equation to (16) in tensor notation is

$$dW = \sigma_{ij}\, d\epsilon_{ij} \quad (i,j = 1, 2, 3).$$

Expression (16) is analogous to the expressions for the work of magnetization, Chapter III, (11), and of polarization, Chapter IV, (9). Each has the form, for unit volume, of a 'force' (H_i, E_i or σ_i) multiplied by a small 'displacement' (dB_i, dD_i or $d\epsilon_i$). Just as in the magnetic and electrical cases we go on to prove that the matrix connecting the two quantities involved, in this case (c_{ij}), is symmetrical.

If the deformation process is isothermal and reversible the work done is equal to the increase in the free energy $d\Psi$ and we may write, per unit volume,

$$d\Psi = dW = \sigma_i\, d\epsilon_i. \tag{17}$$

If Hooke's Law (14) is obeyed this becomes

$$d\Psi = c_{ij}\,\epsilon_j\, d\epsilon_i. \tag{18}$$

Hence,

$$\frac{\partial\Psi}{\partial\epsilon_i} = c_{ij}\,\epsilon_j$$

(the argument is similar to that in Ch. III, § 2). Differentiating both sides of this equation with respect to ϵ_j we have

$$\frac{\partial}{\partial\epsilon_j}\left(\frac{\partial\Psi}{\partial\epsilon_i}\right) = c_{ij}.$$

But since Ψ is a function only of the state of the body, specified by the strain components, the order of differentiation is immaterial, and the left-hand side of this equation is symmetrical with respect to i and j. Hence

$$c_{ij} = c_{ji}. \tag{19}$$

It follows from the form of the relationship between the c_{ij} and the s_{ij} (Ch. IX, § 4.2) that

$$s_{ij} = s_{ji}. \tag{20}$$

The symmetry of the (c_{ij}) and (s_{ij}) matrices implied by relations (19) and (20) further reduces the number of independent stiffness constants and compliances from 36 to 21.

Integrating equation (18) and using (19) we find that the work necessary to produce a strain ϵ_i, called the *strain energy*, is

$$\tfrac{1}{2}c_{ij}\,\epsilon_i\,\epsilon_j \tag{21}$$

per unit volume of the crystal [cf. Ch. III, (17)].

4. The effect of crystal symmetry

The presence of symmetry in the crystal reduces still further the number of independent s_{ij} and c_{ij}. It should first be noticed that elasticity is a centrosymmetrical property. By this it is meant that, if the reference axes are transformed by the operation of a centre of symmetry, the components of s_{ijkl} and c_{ijkl} remain unaltered. The proof is simple. The elements a_{ij} of the transformation matrix are equal to $-\delta_{ij}$. We have, therefore, from (11),

$$s'_{ijkl} = \delta_{im}\,\delta_{jn}\,\delta_{ko}\,\delta_{lp}\,s_{mnop} = s_{ijkl},$$

by the substitution property of δ_{ij} (p. 35); and similarly for c_{ijkl}. However, symmetry elements other than a centre do, in general, impose conditions on the constants, which we must now consider. The methods used for finding the conditions are exactly the same as those used for the piezoelectric moduli.

(i) *Pure symmetry arguments.* The conditions on the s_{ij} and c_{ij} can often be deduced purely from symmetry arguments without analysis, just as with the piezoelectric moduli. Consider the compliance s_{34} in the orthorhombic class *222* for instance. It measures the extension in the Ox_3 direction when the crystal is sheared about the Ox_1 direction, as in Fig. 8.1 *a*. Now operate on the whole system, crystal plus shearing forces, with a diad axis parallel to Ox_2. The crystal remains unaltered since its symmetry includes this diad axis; so does the extension parallel to Ox_3. The forces on the faces, however, are changed to those shown in Fig. 8.1 *b*. We therefore have the same crystal, still extended in the Ox_3 direction, but now under the reverse forces. This situation is only

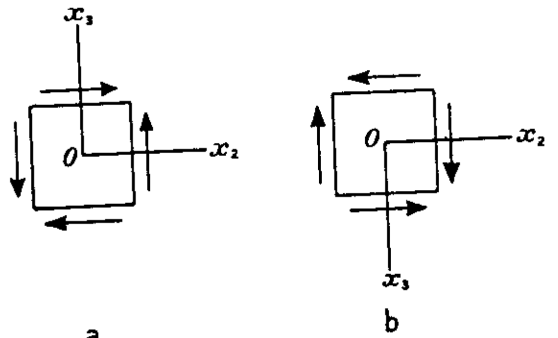


FIG. 8.1. Illustrating that in class *222* the compliance s_{34} vanishes.

possible if the extension is zero. Hence $s_{34} = 0$. Similar symmetry arguments may be framed for most of the compliances in the various classes.

EXERCISE 8.1. Deduce purely from symmetry arguments the form of the compliance matrix for class *4*.

(ii) *Direct inspection method.* The direct inspection method, described on pp. 118–20, gives the quickest way of finding the independent coefficients in all classes except those of the trigonal and hexagonal systems. One example will suffice to illustrate it; we choose class $\bar{4}$.

With the $\bar{4}$ axis parallel to x_3, the axes transform as follows:

$$1 \to 2, \quad 2 \to -1, \quad 3 \to -3.$$

Hence in the *four-suffix notation* the pairs of suffixes transform as follows:

$$11 \to 22, \quad 22 \to 11, \quad 33 \to 33, \quad 23 \to 13, \quad 31 \to -32, \quad 12 \to -21.$$

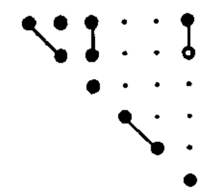
In the *two-suffix notation* these transformations are:

$$5 \to -4, \quad 6 \to -6.$$

The array of suffixes in the matrix written out in the usual order (15) thus transforms to:

$$\begin{array}{cccccc}
22 & 21 & 23 & 25 & -24 & -26 \\
 & 11 & 13 & 15 & -14 & -16 \\
 & & 33 & 35 & -34 & -36 \\
 & & & 55 & -54 & -56 \\
 & & & & 44 & 46 \\
 & & & & & 66
\end{array}$$

We have omitted the lower left-hand half of the array, since the matrix is symmetrical. Equating this array, component by component, with the original one, the relations between the components are at once seen to be



The notation used here has already been described (see the key to Table 9): a light dot denotes a component that is zero; a heavy dot denotes a non-zero component; a line joining two heavy dots means that the two components are numerically equal; a heavy dot and an open circle linked together denote components that are numerically equal, but opposite in sign.

(iii) *Results for all the crystal classes.* For the trigonal and hexagonal systems an analytical method has to be used, just as for the piezoelectric moduli. Direct inspection may be used in all other cases, and the number of independent components may be checked by group theory. The results, for both the s and the c matrices, are given in Table 9. A full key to the notation appears at the head of the table. To illustrate the meaning of the notation we may refer to classes *3* and $\bar{3}$, where

$$s_{15} = -s_{25},\ s_{46} = 2s_{25},\ \text{and}\ s_{66} = 2(s_{11}-s_{12});$$

$$c_{15} = -c_{25},\ c_{46} = c_{25},\ \text{and}\ c_{66} = \tfrac{1}{2}(c_{11}-c_{12}).$$

The number of independent components is given in brackets after each matrix. The orientation of the axes conforms to the conventions in *Standards on piezoelectric crystals* (1949) (Appendix B, p. 282) except where otherwise stated.

TABLE 9

Form of the (s_{ij}) *and* (c_{ij}) *matrices*

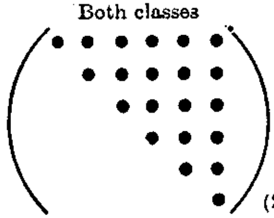
KEY TO NOTATION

	·	zero component
	●	non-zero component
	●—●	equal components
	●—○	components numerically equal, but opposite in sign
For s	◉	twice the numerical equal of the heavy dot component to which it is joined
For c	◉	the numerical equal of the heavy dot component to which it is joined
For s	×	$2(s_{11}-s_{12})$
For c	×	$\frac{1}{2}(c_{11}-c_{12})$

All the matrices are symmetrical about the leading diagonal.

TRICLINIC

Both classes

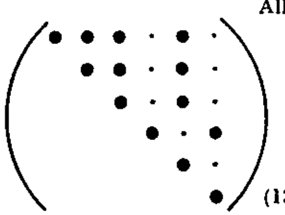


(21)

MONOCLINIC

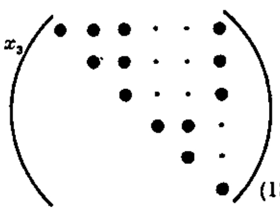
All classes

Diad $\parallel x_2$ (standard orientation)



(13)

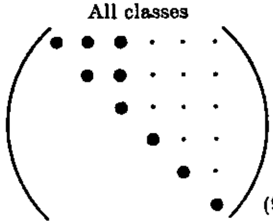
Diad $\parallel x_3$



(13)

ORTHORHOMBIC

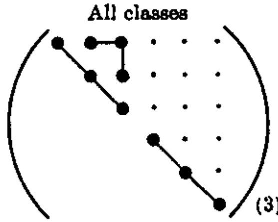
All classes



(9)

CUBIC

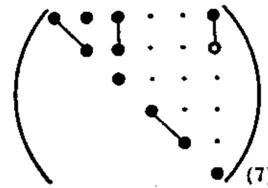
All classes



(3)

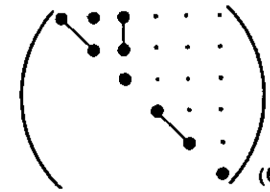
TETRAGONAL

Classes $4, \bar{4}, 4/m$



(7)

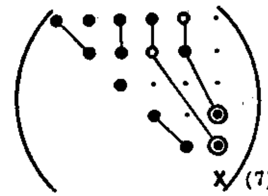
Classes $4mm, \bar{4}2m,$† $422, 4/mmm$



(6)

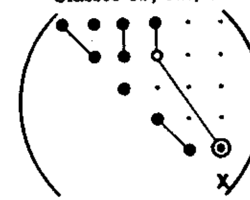
TRIGONAL

Classes $3, \bar{3}$



(7)

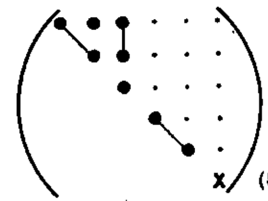
Classes $32, \bar{3}m, 3m$



(6)

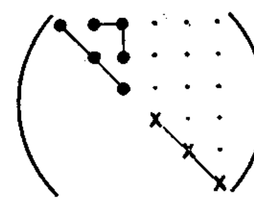
HEXAGONAL

All classes



(5)

ISOTROPIC



(2)

The form of the matrix given in the table for a completely isotropic material is obtained from the cubic matrix by requiring that the components should be unaltered by rotations of 45° about the reference axes. It may be verified that the form so given is unaltered by any rotation of axes. For second-rank tensor properties cubic crystals were

† The same matrix holds for both possible orientations of class $\bar{4}2m$ ($2 \parallel x_1$ and $m \perp x_1$) since the addition of a centre of symmetry makes the two orientations indistinguishable

found to be isotropic. We now see that the elastic properties of cubic crystals, given by fourth-rank tensors, are not isotropic.

(iv) *Numerical example.* For a numerical example we choose ammonium dihydrogen phosphate (ADP) (class $\bar{4}2m$). This was the crystal for which piezoelectric data were given on p. 120. The components of the (s_{ij}) and (c_{ij}) matrices in m.k.s. units† for ADP at 0° C are measured (Mason 1946) as:

$$(s_{ij}) = \begin{pmatrix} 1\cdot8 & 0\cdot7 & -1\cdot1 & 0 & 0 & 0 \\ & 1\cdot8 & -1\cdot1 & 0 & 0 & 0 \\ & & 4\cdot3 & 0 & 0 & 0 \\ & & & 11\cdot3 & 0 & 0 \\ & & & & 11\cdot3 & 0 \\ & & & & & 16\cdot2 \end{pmatrix} \times 10^{-11},$$

$$(c_{ij}) = \begin{pmatrix} 0\cdot71 & -0\cdot20 & 0\cdot13 & 0 & 0 & 0 \\ & 0\cdot71 & 0\cdot13 & 0 & 0 & 0 \\ & & 0\cdot30 & 0 & 0 & 0 \\ & & & 0\cdot088 & 0 & 0 \\ & & & & 0\cdot088 & 0 \\ & & & & & 0\cdot070 \end{pmatrix} \times 10^{11}.$$

4.1. Further restrictions on the constants. The strain energy of a crystal given by (21) must be positive, for otherwise the crystal would be unstable. This means that the quadratic form (21) must be positive definite, that is, greater than zero for all real values of the ϵ_{ij} unless all the ϵ_{ij} are zero. This implies further restrictions on the s_{ij} and c_{ij}, which may be found by standard algebraical methods; see, for example, the book by Ferrar (1941), p. 138.

For a hexagonal crystal these restrictions on the c_{ij} are:

$$c_{44} > 0, \quad c_{11} > |c_{12}|, \quad (c_{11}+c_{12})c_{33} > 2c_{13}^2.$$

For a cubic crystal:

$$c_{44} > 0, \quad c_{11} > |c_{12}|, \quad c_{11}+2c_{12} > 0.$$

The compliances s_{ij} are subject to identical restrictions.

4.2. Stress-strain relations for isotropic materials. Using the (s_{ij}) matrix given in Table 9 for an isotropic material, we may express the s_{ij} in terms of more familiar quantities, such as Young's Modulus and the Rigidity Modulus. First we write out the equations for the strain components in terms of the stress components, and, by their

† ... m²/newton respectively.

side, for comparison, we write the same equations in a form frequently used in elasticity textbooks:

$$\left.\begin{aligned} \epsilon_1 &= s_{11}\sigma_1+s_{12}\sigma_2+s_{12}\sigma_3 \\ \epsilon_2 &= s_{12}\sigma_1+s_{11}\sigma_2+s_{12}\sigma_3 \\ \epsilon_3 &= s_{12}\sigma_1+s_{12}\sigma_2+s_{11}\sigma_3 \\ \epsilon_4 &= 2(s_{11}-s_{12})\sigma_4 \\ \epsilon_5 &= 2(s_{11}-s_{12})\sigma_5 \\ \epsilon_6 &= 2(s_{11}-s_{12})\sigma_6 \end{aligned}\right\}; \qquad \left.\begin{aligned} \epsilon_1 &= \frac{1}{E}\{\sigma_1-\nu(\sigma_2+\sigma_3)\} \\ \epsilon_2 &= \frac{1}{E}\{\sigma_2-\nu(\sigma_3+\sigma_1)\} \\ \epsilon_3 &= \frac{1}{E}\{\sigma_3-\nu(\sigma_1+\sigma_2)\} \\ \epsilon_4 &= \frac{1}{G}\sigma_4 \\ \epsilon_5 &= \frac{1}{G}\sigma_5 \\ \epsilon_6 &= \frac{1}{G}\sigma_6 \end{aligned}\right\}.$$

E is Young's Modulus, G is the Rigidity Modulus and ν is Poisson's Ratio. Comparing coefficients we have

$$s_{11} = 1/E, \quad s_{12} = -\nu/E \quad \text{and} \quad 2(s_{11}-s_{12}) = 1/G, \tag{22}$$

from which follows the relation

$$G = E/\{2(1+\nu)\}.$$

The equations for the stresses in terms of the strains, using the stiffness constants, may be compared with the same equations as they are usually written in books on elasticity in the λ, μ notation. Thus:

$$\left.\begin{aligned} \sigma_1 &= c_{11}\epsilon_1+c_{12}\epsilon_2+c_{12}\epsilon_3 \\ \sigma_2 &= c_{12}\epsilon_1+c_{11}\epsilon_2+c_{12}\epsilon_3 \\ \sigma_3 &= c_{12}\epsilon_1+c_{12}\epsilon_2+c_{11}\epsilon_3 \\ \sigma_4 &= \tfrac{1}{2}(c_{11}-c_{12})\epsilon_4 \\ \sigma_5 &= \tfrac{1}{2}(c_{11}-c_{12})\epsilon_5 \\ \sigma_6 &= \tfrac{1}{2}(c_{11}-c_{12})\epsilon_6 \end{aligned}\right\}; \qquad \left.\begin{aligned} \sigma_1 &= (2\mu+\lambda)\epsilon_1+\lambda\epsilon_2+\lambda\epsilon_3 \\ \sigma_2 &= \lambda\epsilon_1+(2\mu+\lambda)\epsilon_2+\lambda\epsilon_3 \\ \sigma_3 &= \lambda\epsilon_1+\lambda\epsilon_2+(2\mu+\lambda)\epsilon_3 \\ \sigma_4 &= \mu\epsilon_4 \\ \sigma_5 &= \mu\epsilon_5 \\ \sigma_6 &= \mu\epsilon_6 \end{aligned}\right\};$$

whence $\quad c_{11} = 2\mu+\lambda \quad \text{and} \quad c_{12} = \lambda.$

5. Representation surfaces and Young's Modulus

No single surface can represent the elastic behaviour of a crystal completely. A surface that is useful in practice is one that shows the variation of Young's Modulus with direction. A bar of the crystal is supposed to be cut with its length parallel to some arbitrary direction, Ox_1', and loaded in simple tension. As we have seen, the tension prostrains as well. The Young's Modulus for the direction of the tension is defined as the ratio of the longitudinal stress to the longitudinal strain, that is, $1/s'_{11}$. Surfaces for which the radius vector in the direction Ox'_1 is proportional either to s'_{11} or to $1/s'_{11}$ are commonly used.

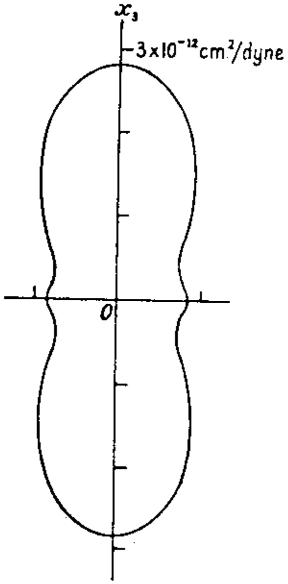


FIG. 8.2. A central section of a representation surface for Young's Modulus in zinc. The length of the radius vector is proportional to s'_{11}, that is, to the reciprocal of Young's Modulus. (After Goens 1933.)

As an example we may take a crystal of zinc (hexagonal). Writing

$$s'_{1111} = a_{1m}\,a_{1n}\,a_{1p}\,a_{1q}\,s_{mnpq},$$

and using the matrix of compliances given in Table 9, we find, after some reduction and changing to the contracted notation,

$$s'_{11} = s_{11}(1-a_{13}^2)^2+s_{33}\,a_{13}^4+(s_{44}+2s_{13})(1-a_{13}^2)a_{13}^2,$$

or

$$s'_{11} = s_{11}\sin^4\theta+s_{33}\cos^4\theta+(s_{44}+2s_{13})\sin^2\theta\cos^2\theta, \tag{23}$$

where θ is the angle between the arbitrary direction Ox'_1 and the crystallographic z-axis, Ox_3. The s'_{11} or $1/s'_{11}$ surface is then one of revolution about Ox_3. The compliances are (Goens 1933):

$$s_{11} = 8{\cdot}4, \quad s_{12} = 1{\cdot}1, \quad s_{13} = -7{\cdot}8,$$
$$s_{33} = 28{\cdot}7, \quad s_{44} = 26{\cdot}4\times10^{-12}\ \mathrm{m^2/newton};†$$

a section of the s'_{11} surface is shown in Fig. 8.2.

We collect here the expressions for the reciprocal of Young's Modulus in the direction of the unit vector l_i in the various crystal systems.

Triclinic system

$$\begin{aligned} l_1^4 s_{11}+2l_1^2 l_2^2 s_{12}+2l_1^2 l_3^2 s_{13}+(2l_1^2 l_2 l_3 s_{14})+\ & 2l_1^3 l_3 s_{15}+(2l_1^3 l_2 s_{16})+\\ +\ l_2^4 s_{22}+2l_2^2 l_3^2 s_{23}+(2l_2^3 l_3 s_{24})+\ & 2l_1 l_2^2 l_3 s_{25}+(2l_1 l_2^3 s_{26})+\\ +\ l_3^4 s_{33}+(2l_2 l_3^3 s_{34})+\ & 2l_1 l_3^3 s_{35}+(2l_1 l_2 l_3^2 s_{36})+\\ +\ & l_2^2 l_3^2 s_{44}+(2l_1 l_2 l_3^2 s_{45})+2l_1 l_3^2 l_2 s_{46}+\\ & +\ l_1^2 l_3^2 s_{55}+(2l_1^2 l_2 l_3 s_{56})+\\ & +\ l_1^2 l_2^2 s_{66}. \end{aligned}$$

Monoclinic system (standard orientation, p. 284).

As above but omitting terms in brackets

† 10 m²/newton = 1 cm²/dyne.

Orthorhombic system

$$\begin{aligned} & l_1^4 s_{11}+2l_1^2 l_2^2 s_{12}+2l_1^2 l_3^2 s_{13}+\\ & +\ l_2^4 s_{22}+2l_2^2 l_3^2 s_{23}+\\ & \qquad +\ l_3^4 s_{33}+l_2^2 l_3^2 s_{44}+l_1^2 l_3^2 s_{55}+l_1^2 l_2^2 s_{66}. \end{aligned}$$

Tetragonal system. *Classes 4, $\bar{4}$, 4/m*

$$(l_1^4+l_2^4)s_{11}+l_3^4 s_{33}+l_1^2 l_2^2(2s_{12}+s_{66})+l_3^2(1-l_3^2)(2s_{13}+s_{44})+[2l_1 l_2(l_1^2-l_2^2)s_{16}].$$

Classes 4mm, $\bar{4}$2m, 422, 4/mmm

As above but omitting the term in square brackets.

Cubic system

$$s_{11}-2(s_{11}-s_{12}-\tfrac{1}{2}s_{44})(l_1^2 l_2^2+l_2^2 l_3^2+l_3^2 l_1^2).$$

Trigonal system. *Classes 3, $\bar{3}$*

$$(1-l_3^2)^2 s_{11}+l_3^4 s_{33}+l_3^2(1-l_3^2)(2s_{13}+s_{44})+2l_2 l_3(3l_1^2-l_2^2)s_{14}+[2l_1 l_3(3l_2^2-l_1^2)s_{25}].$$

Classes 3m, 32, $\bar{3}$m

As above but omitting the term in square brackets.

Hexagonal system

$$(1-l_3^2)^2 s_{11}+l_3^4 s_{33}+l_3^2(1-l_3^2)(2s_{13}+s_{44}).$$

Notice that in the cubic system Young's Modulus is not isotropic. The variation with direction depends on $(l_1^2 l_2^2+l_2^2 l_3^2+l_3^2 l_1^2)$. This quantity is zero for the directions of the cube axes $\langle 100\rangle$ and has its maximum value of $\frac{1}{3}$ in the $\langle 111\rangle$ directions. Hence, if $(s_{11}-s_{12}-\frac{1}{2}s_{44})$ is positive (as it is for all cubic metals except molybdenum), Young's Modulus is a maximum in the $\langle 111\rangle$ directions and a minimum in the $\langle 100\rangle$ directions. A surface for which the radius vector is directly proportional to Young's Modulus would then have the form of a cube with rounded corners and depressions at the centres of the faces. Its central sections normal to $\langle 111\rangle$ are readily seen to be circles. $(s_{11}-s_{12}-\frac{1}{2}s_{44}) = 0$ is the condition for elastic isotropy. If $(s_{11}-s_{12}-\frac{1}{2}s_{44})$ is negative, Young's Modulus is a minimum for $\langle 111\rangle$ and a maximum for $\langle 100\rangle$. The Young's Modulus surface then has protuberances along the cube axes.†

6. Volume and linear compressibility of a crystal

(i) *Volume compressibility.* We calculate the proportional decrease in volume of a crystal when subjected to unit hydrostatic pressure, that is, its *volume compressibility*. In equation (1) put $\sigma_{kl} = -p\delta_{kl}$. Then

$$\epsilon_{ij} = -ps_{ijkl}\delta_{kl} = -ps_{ijkk}. \tag{24}$$

† Representation surfaces both for Young's Modulus and for the twisting modulus of a cylinder are illustrated in the books by Wooster (1938) and by Schmid and Boas (1950).

For the dilation Δ (p. 100) we have

$$\Delta = \epsilon_{ii} = -ps_{iikk};$$

and so the volume compressibility, $-\Delta/p$, is s_{iikk}. This is another example of an invariant formed from a tensor. In the matrix notation the volume compressibility is

$$s_{11}+s_{22}+s_{33}+2(s_{12}+s_{23}+s_{31}), \tag{25}$$

and is thus the sum of the nine coefficients in the upper left-hand corner of the compliance matrix. For a cubic crystal it is evidently $3(s_{11}+2s_{12})$. This last expression also holds for an isotropic material; but for this case it is customary to define the reciprocal of the volume compressibility as the *Bulk Modulus*

$$K = 1/\{3(s_{11}+2s_{12})\} = E/\{3(1-2\nu)\},$$

from equations (22).

(ii) *Linear compressibility*. The linear compressibility of a crystal is the relative decrease in length of a line when the crystal is subjected to unit hydrostatic pressure. In general it varies with direction. Under pressure p the stretch of a line in the direction of the unit vector l_i is (p. 101)

$$\epsilon_{ij} l_i l_j = -ps_{ijkk} l_i l_j$$

from (24), and so the linear compressibility is

$$\beta = s_{ijkk} l_i l_j. \tag{26}$$

Written out in the matrix notation for the seven crystal systems the expressions for β are:

Triclinic system

$$\beta = (s_{11}+s_{12}+s_{13})l_1^2+(s_{16}+s_{26}+s_{36})l_1 l_2+(s_{15}+s_{25}+s_{35})l_3 l_1+ \\ +(s_{12}+s_{22}+s_{23})l_2^2 \quad +(s_{14}+s_{24}+s_{34})l_2 l_3+ \\ +(s_{13}+s_{23}+s_{33})l_3^2.$$

Monoclinic system (standard orientation, p. 284)

$$\beta = (s_{11}+s_{12}+s_{13})l_1^2+(s_{12}+s_{22}+s_{23})l_2^2+(s_{13}+s_{23}+s_{33})l_3^2+ \\ +(s_{15}+s_{25}+s_{35})l_3 l_1.$$

Orthorhombic system

$$\beta = (s_{11}+s_{12}+s_{13})l_1^2+(s_{12}+s_{22}+s_{23})l_2^2+(s_{13}+s_{23}+s_{33})l_3^2.$$

Tetragonal, trigonal and hexagonal systems. (*All classes*)

$$\beta = (s_{11}+s_{12}+s_{13})-(s_{11}+s_{12}-s_{13}-s_{33})l_3^2.$$

Cubic system

$$\beta = s_{11}+2s_{12}.$$

Thus, the linear compressibility in the optically uniaxial systems is

rotationally symmetrical about the unique axis. In the cubic system the linear compressibility is isotropic: a sphere of a cubic crystal under hydrostatic pressure remains a sphere.

Exercise 8.2. Prove that the volume change of a cubic crystal under uniaxial tension T is independent of the direction of the tension and is given by $(s_{11}+2s_{12})T$.

7. Relations between the compliances and the stiffnesses

Explicit general equations for the s_{ij} in terms of the c_{ij} and vice versa are derived in the next chapter. We give here a number of useful relations between the s_{ij} and the c_{ij} in some of the more symmetrical classes (Boas and Mackenzie 1950):

Trigonal system. *Classes 3m, 32,* $\bar{3}m$

$$c_{11}+c_{12} = s_{33}/s, \quad c_{11}-c_{12} = s_{44}/s', \quad c_{13} = -s_{13}/s,$$
$$c_{14} = -s_{14}/s', \quad c_{33} = (s_{11}+s_{12})/s, \quad c_{44} = (s_{11}-s_{12})/s',$$

where
$$s = s_{33}(s_{11}+s_{12})-2s_{13}^2,$$
and
$$s' = s_{44}(s_{11}-s_{12})-2s_{14}^2.$$

Tetragonal system. *Classes 4mm,* $\bar{4}2m$*, 422, 4/mmm*

$$c_{11}+c_{12} = s_{33}/s, \quad c_{11}-c_{12} = 1/(s_{11}-s_{12}), \quad c_{13} = -s_{13}/s,$$
$$c_{33} = (s_{11}+s_{12})/s, \quad c_{44} = 1/s_{44}, \quad c_{66} = 1/s_{66},$$

where
$$s = s_{33}(s_{11}+s_{12})-2s_{13}^2.$$

Hexagonal system. (*All classes*)

$$c_{11}+c_{12} = s_{33}/s, \quad c_{11}-c_{12} = 1/(s_{11}-s_{12}), \quad c_{13} = -s_{13}/s,$$
$$c_{33} = (s_{11}+s_{12})/s, \quad c_{44} = 1/s_{44},$$

where
$$s = s_{33}(s_{11}+s_{12})-2s_{13}^2.$$

Cubic system. (*All classes*)

$$c_{11} = \frac{s_{11}+s_{12}}{(s_{11}-s_{12})(s_{11}+2s_{12})},$$
$$c_{12} = \frac{-s_{12}}{(s_{11}-s_{12})(s_{11}+2s_{12})},$$
$$c_{44} = 1/s_{44}.$$

8. Numerical values of the elastic coefficients

Some further numerical data on the elastic compliances of crystals are given in Table 10. Many cubic crystals are markedly anisotropic in their elastic behaviour, but tungsten and aluminium are only slightly anisotropic. With hexagonal crystals the anisotropy is sometimes great: for example, the linear compressibility of zinc is $1{\cdot}31\times10^{-11}$ m²/newton parallel to the z-axis and $0{\cdot}175\times10^{-11}$ in all directions perpendicular to it (as may be calculated from the expression for β on p. 146). For

Thus, the linear compressibility in the optically uniaxial systems is

cadmium the linear compressibilities are 1·69 and $0{\cdot}15\times10^{-13}$. For tellurium, which has a chain structure, the linear compressibility parallel to the chain axis is negative.

TABLE 10

Elasticity of Crystals

Compliances at room temperature (unit = 10^{-11} m²/newton)

Crystal	*Class*	s_{11}	s_{12}	s_{44}	s_{33}	s_{13}	s_{14}	s_{66}
Sodium chloride	$m3m$	2·21	−0·45	7·83	..	..	..	..
Aluminium	$m3m$	1·59	−0·58	3·52	..	..	..	..
Copper	$m3m$	1·49	−0·63	1·33	..	..	..	..
Nickel	$m3m$	0·799	−0·312	0·844	..	..	..	..
Tungsten	$m3m$	0·257	−0·073	0·660	..	..	..	..
Sodium chlorate	23	2·2	−0·6	8·6	..	..	..	..
Tin	$4/mmm$	1·85	−0·99	5·70	1·18	−0·25	..	13·5
ADP	$\bar{4}2m$	1·8	0·7	11·3	4·3	−1·1	..	16·2
Zinc	$6/mmm$	0·84	0·11	2·64	2·87	−0·78	..	..
Cadmium	$6/mmm$	1·23	−0·15	5·40	3·55	−0·93	..	..
Quartz	32	1·27	−0·17	2·01	0·97	−0·15	−0·43	..
Tourmaline	$3m$	0·40	−0·10	1·51	0·63	−0·016	0·058	..

Values are taken from the following sources: Boas and Mackenzie (1950), Van Dyke and Gordon (1950), and Hearmon (1946); the last of these contains a full collection of elastic data up to the end of 1944, and has been supplemented to cover the period 1945–1955 (Hearmon 1956). Other recent collections are those of Bhagavantam (1955) and Raman and Krishnamurti (1955). See also the important review article by Huntington (1958), which deals with the microscopic, atomic, theory of elastic constants as well as with the macroscopic theory.

SUMMARY

Hooke's Law for a crystal is written

$$\epsilon_{ij} = s_{ijkl}\sigma_{kl}; \quad \sigma_{ij} = c_{ijkl}\epsilon_{kl}, \qquad (1);\ (2)$$

where the s_{ijkl} and c_{ijkl} are the components of fourth-rank tensors. The s_{ijkl} are the *elastic compliances* and the c_{ijkl} are the *elastic stiffnesses*. Since σ_{ij} in these equations is the symmetric part of the stress tensor it is convenient to put

$$s_{ijkl} = s_{ijlk}.$$

The symmetry of the strain tensor implies that

$$s_{ijkl} = s_{jikl}.$$

c_{ijkl} is likewise symmetrical in the first two and the last two suffixes. These relations reduce the number of independent s's and c's to 36.

Matrix notation. We abbreviate the first two and the last two suffixes of s_{ijkl} and c_{ijkl} into single suffixes running from 1 to 6, as with the piezoelectric moduli. We introduce factors of 2 and 4 in the s_{ijkl} but not in the c_{ijkl}. We use also the single suffix notation already developed for the stress and strain components. Then (1) and (2) become

$$\epsilon_i = s_{ij}\sigma_j; \qquad \sigma_i = c_{ij}\epsilon_j \quad (i,j = 1,2,\ldots,6). \qquad (13);\ (14)$$

(s_{ij}) and (c_{ij}) are 6-row, 6-column matrices.

to it (as may be calculated from the expression [illegible]

Other results. The work done per unit volume when there is a small change of strain in a crystal is

$$dW = \sigma_i\, d\epsilon_i. \qquad (16)$$

When the change is isothermal and reversible dW may be equated with the increase in free energy $d\Psi$. The fact that $d\Psi$ is a perfect differential, together with (14), then implies that

$$c_{ij} = c_{ji}, \qquad s_{ij} = s_{ji}; \qquad (19),\ (20)$$

and the strain energy per unit volume is

$$\tfrac{1}{2}c_{ij}\epsilon_i\epsilon_j. \qquad (21)$$

(19) and (20) reduce the number of independent compliances and stiffnesses from 36 to 21, and the number is further reduced by crystal symmetry. The condition that (21) should be positive definite imposes still further restrictions.

A crystal bar loaded in simple tension undergoes longitudinal and lateral strains and also shear strains. *Young's Modulus* for the direction of the tension is defined as the ratio of the longitudinal tension to the longitudinal strain. Young's Modulus is anisotropic for all crystal classes, including the cubic classes, and its variation with direction may be represented by a surface.

Let unit hydrostatic pressure be applied to a crystal. Then: (1) the proportional decrease in volume, the *volume compressibility*, is s_{iikk} and is the sum of the nine components in the upper left-hand corner of the (s_{ij}) matrix; (2) the proportional decrease in length of a line in the crystal in the direction l_i, the ***linear compressibility***, is $\beta = s_{ijkk}l_i l_j$. The linear compressibility of cubic crystals is isotropic.